

Arithmeticity of complex hyperbolic triangle groups

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Abstract

Complex hyperbolic triangle groups, originally studied by Mostow in building the first nonarithmetic lattices in $\mathrm{PU}(2,1)$, are a natural generalization of the classical triangle groups. A theorem of Takeuchi states that there are only finitely many Fuchsian triangle groups that determine an arithmetic lattice in $\mathrm{PSL}_2(\mathbb{R})$, so triangle groups are generically nonarithmetic. We prove similar finiteness theorems for complex hyperbolic triangle groups that determine an arithmetic lattice in $\mathrm{PU}(2,1)$.

1 Introduction

In his seminal 1980 paper, Mostow constructed lattices in $\mathrm{PU}(2,1)$ generated by three complex reflections [10]. He not only gave a new geometric method for building lattices acting on the complex hyperbolic plane, but gave the first examples of nonarithmetic lattices in $\mathrm{PU}(2,1)$. Complex reflection groups are a generalization of groups generated by reflections through hyperplanes in constant curvature spaces, and Mostow's groups are a natural extension to the complex hyperbolic plane of the classical triangle groups. They are often called *complex hyperbolic triangle groups*. We introduce these groups in §2. See also [5, 17], which, along with [10], inspired much of the recent surge of activity surrounding these groups.

Around the same time, Takeuchi classified the Fuchsian triangle groups that determine arithmetic lattices in $\mathrm{PSL}_2(\mathbb{R})$ [18]. In particular, he proved that there are finitely many and gave a complete list. Since there are infinitely many triangle groups acting on the hyperbolic plane discretely with

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finite covolume, triangle groups are generically nonarithmetic. The purpose of this paper is to give analogous finiteness results for complex hyperbolic triangle groups that determine an arithmetic lattice in $\mathrm{PU}(2, 1)$.

A particular difficulty with complex hyperbolic triangle groups is that the complex triangle is not uniquely determined by its angles. One must also consider the so-called *angular invariant* $\psi \in [0, 2\pi)$. See §2. In particular, there is a 1-dimensional deformation space of complex triangles with fixed triple of angles. The typical assumption is that ψ is a rational multiple of π , in which case the angular invariant is called *rational*. We call it *irrational* otherwise.

When a complex hyperbolic triangle group is also an arithmetic lattice, we will call it an arithmetic complex hyperbolic triangle group. Note that this immediately implies discreteness. Our first result is for nonuniform arithmetic complex hyperbolic triangle groups. We prove the following in §6.

Theorem 1.1. *There are finitely many nonuniform arithmetic complex hyperbolic triangle groups with rational angular invariant. If Γ is a nonuniform arithmetic complex hyperbolic triangle group with irrational angular invariant ψ , then $e^{i\psi}$ is contained in a biquadratic extension of \mathbb{Q} .*

We emphasize that complex reflection groups are allowed to have generators of arbitrary finite order. A usual assumption is that all generators have the same order, a restriction that we avoid. See Theorem 6.1 for a more precise formulation of Theorem 1.1. Proving that a candidate is indeed a lattice is remarkably difficult, as evidenced in [10, 3], so we do not give a definitive list. One consequence of the proof (see Theorem 1.5(1) below) is the following.

Corollary 1.2. *Suppose that Γ is a nonuniform lattice in $\mathrm{U}(2, 1)$. If Γ contains a complex reflection of order 5 or at least 7, then Γ is nonarithmetic.*

In the cocompact setting, the arithmetic is much more complicated. Arithmetic subgroups of $\mathrm{U}(2, 1)$ come in two types, defined in §3, often called first and second. In §4 we prove the following auxiliary result, generalizing a well-known fact for hyperbolic reflection groups.

Theorem 1.3. *Let $\Gamma < \mathrm{U}(2, 1)$ be a lattice containing a complex reflection. Then Γ contains a Fuchsian subgroup stabilizing the wall of the reflection in $\mathbf{H}_{\mathbb{C}}^2$.*

We also give a generalization to higher-dimensional complex reflection groups. Theorem 1.3 leads to the following, which we also prove in §4.

Theorem 1.4. *Let $\Gamma < \mathrm{U}(2, 1)$ be a lattice, and suppose that Γ is commensurable with a lattice Λ containing a complex reflection. Then Γ is either arithmetic of first type or nonarithmetic.*

In particular, when considering a complex reflection group as a candidate for a nonarithmetic lattice, one must only show that it is not of the first type. Fortunately, this is the case where the arithmetic is simplest to understand.

The effect of the angular invariant is a particular sticking point in generalizing Takeuchi's methods. In §5, the technical heart of the paper, we study the interdependence between the geometric invariants of the triangle and the arithmetic of the lattice. We collect the most useful of these facts as the following. See §§2-3 for our notation.

Theorem 1.5. *Suppose that Γ is an arithmetic complex hyperbolic triangle group. Suppose that for $j = 1, 2, 3$ the generators have reflection factors η_j , the complex angles of the triangle are θ_j , and that the angular invariant is ψ . Let E be the totally imaginary quadratic extension of the totally real field F that defines Γ as an arithmetic lattice. Then:*

1. $\eta_j \in E$ for all j ;
2. $\cos^2 \theta_j \in F$ for all j ;
3. $e^{2i\psi} \in E$ and $\cos^2 \psi \in F$;
4. If $\theta_j \leq \pi/3$ for all j , then

$$\cos^2 \psi \in \mathbb{Q}(\cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, \cos \theta_1 \cos \theta_2 \cos \theta_3);$$

5. $E \subseteq \mathbb{Q}(\cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, e^{i\psi} \cos \theta_1 \cos \theta_2 \cos \theta_3)$;
6. If ψ is rational, then E is a subfield of a cyclotomic field.

In §6, we use the results from §5 to prove finiteness results for cocompact arithmetic complex hyperbolic triangle groups with rational angular invariant. We also give restrictions for irrational angular invariants, though it is unknown whether such a lattice exists. When the complex triangle is a right triangle, we prove the following.

Theorem 1.6. *Suppose that Γ is an arithmetic complex hyperbolic triangle group for which the associated complex triangle is a right triangle. Then the angles of the triangle are the angles of an arithmetic Fuchsian triangle group. There are finitely many such Γ with rational angular invariant.*

Finally, we consider equilateral triangles at the end of §6. This is the case which has received the most attention, in particular from Mostow [10] and, in the ideal case, by Goldman–Parker [5] and Schwartz [17]. See also [2]. Here we cannot explicitly bound orders of generators, angles, or angular invariants because our proof relies on asymptotic number theory for which we do not know precise constants. Nevertheless, we obtain finiteness in the situation that has received the greatest amount of attention since Mostow’s original paper. See [11, 12, 13, 3] and references therein for more recent examples of lattices and restrictions on discreteness.

Theorem 1.7. *There are finitely many arithmetic complex hyperbolic equilateral triangle groups with rational angular invariant.*

2 Complex hyperbolic triangle groups

We assume some basic knowledge of complex hyperbolic geometry, e.g., the first three chapters of [4]. Let V be a three-dimensional complex vector space, equipped with a hermitian form h of signature $(2, 1)$. Complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^2$ is the space of h -negative lines in V . The metric on $\mathbf{H}_{\mathbb{C}}^2$ is defined via h as in [4, Chapter 3], and the action of $U(2, 1)$ on $\mathbf{H}_{\mathbb{C}}^2$ by isometries descends from its action on V and factors through projection onto $PU(2, 1)$. Its ideal boundary $\partial\mathbf{H}_{\mathbb{C}}^2$ is the space of h -isotropic lines, and we set $\overline{\mathbf{H}}_{\mathbb{C}}^2 = \mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$.

A *complex reflection* is a diagonalizable linear map $R : V \rightarrow V$ with one eigenvalue of multiplicity 2 (or, more generally, multiplicity $n - 1$ when $\dim(V) = n$). We assume that R has finite order, so the third eigenvalue is a root of unity η . We call η the *reflection factor* of R . Decompose $V = V_1 \oplus V_{\eta}$ into the 1- and η -eigenspaces, and choose $v_{\eta} \in V$ so that $V_{\eta} = \text{Span}_{\mathbb{C}}\{v_{\eta}\}$. We begin with an elementary lemma that will be of use later, keeping in mind that every complex reflection has 1 as an eigenvalue.

Lemma 2.1. *Let $A \in GL_n(\mathbb{C})$ be a diagonalizable linear transformation. Let $E \subseteq \mathbb{C}$ be a subfield, and suppose that E^n has a basis consisting of eigenvectors for A . Furthermore, suppose that A has at least one eigenvalue in E and that there exists $x \in \mathbb{C}^{\times}$ so that $xA \in GL_n(E)$. Then all eigenvalues of A are in E .*

Proof. Let $v_1, \dots, v_n \in E^n$ be a basis of eigenvectors for A , and let λ_j be the eigenvalue associated with v_j , $1 \leq j \leq n$. Without loss of generality, $\lambda_1 \in E$. Then xA also has eigenvectors v_1, \dots, v_n , and $xA v_j = x \lambda_j v_j \in E^n$

for all j , since $xA \in \mathrm{GL}_n(E)$. Then $x\lambda_j \in E$, $1 \leq j \leq n$. Since $\lambda_1 \in E$, it follows that $x \in E$, which implies that $\lambda_j \in E$ for all j . \square

Assume that $R \in \mathrm{U}(2, 1)$. Then the fixed point set of R acting on $\mathbf{H}_{\mathbb{C}}^2$ is the subset of h -negative lines in V_1 . This is a totally geodesic holomorphic embedding of the hyperbolic plane if and only if V_{η} is an h -positive line. These subspaces are called *complex hyperbolic lines*. Following [4, §3.1], we call v_{η} a *polar vector* for R .

When V_{η} is h -negative, the fixed set of R on $\mathbf{H}_{\mathbb{C}}^2$ is a point, and R is sometimes called a reflection through that point. The complex reflections in this paper will always be of through complex hyperbolic lines. That is, the η -eigenspace will always be an h -positive line.

Let W be the complex hyperbolic line in $\mathbf{H}_{\mathbb{C}}^2$ fixed by R . We call this the *wall* of R . If v_{η} is a polar vector, then R is the linear transformation

$$z \mapsto z + (\eta - 1) \frac{h(z, v_{\eta})}{h(v_{\eta}, v_{\eta})} v_{\eta}. \quad (1)$$

We refrain from normalizing the polar vector to have h -norm one, since we will often choose a polar vector with coordinates in a subfield E of \mathbb{C} , and $E^3 \subset V$ might not contain an h -norm one representative for the given line of polar vectors.

Now, consider three complex reflections $R_1, R_2, R_3 \in \mathrm{U}(2, 1)$ with respective distinct walls W_1, W_2, W_3 in $\mathbf{H}_{\mathbb{C}}^2$. If v_j is a polar vector for R_j , then W_j and W_{j+1} (with cyclic indices) meet in $\mathbf{H}_{\mathbb{C}}^2$ if and only if

$$h(W_j, W_{j+1}) = \frac{|h(v_j, v_{j+1})|^2}{h(v_j, v_j)h(v_{j+1}, v_{j+1})} < 1, \quad (2)$$

The two walls meet at a point z_j stabilized by the subgroup of $\mathrm{U}(2, 1)$ generated by R_j and R_{j+1} . The *complex angle* θ_j between W_j and W_{j+1} , the minimum angle between the two walls, satisfies $\cos^2 \theta_j = h(W_j, W_{j+1})$.

The walls W_j and W_{j+1} meet at a point p_j in $\partial\mathbf{H}_{\mathbb{C}}^2$ if and only if

$$\frac{|h(v_j, v_{j+1})|^2}{h(v_j, v_j)h(v_{j+1}, v_{j+1})} = 1, \quad (3)$$

so we say that the complex angle is zero. The group generated by R_j and R_{j+1} fixes p_j , so it is contained in a parabolic subgroup of $\mathrm{U}(2, 1)$. See [4, §3.3.2].

Let $\{R_j\}$ be reflections through walls $\{W_j\}$, $j = 1, 2, 3$. When the pairwise intersections of the walls are nontrivial in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, they determine a *complex*

triangle in $\mathbf{H}_{\mathbb{C}}^2$, possibly with ideal vertices. The subgroup $\Delta(R_1, R_2, R_3)$ of $U(2, 1)$ generated by the R_j s is called a *complex hyperbolic triangle group*.

A complex hyperbolic triangle group is sometimes defined as one with order two generators, and groups with higher order generators are called *generalized* triangle groups. We avoid this distinction and do not make the usual assumption that all generators have the same order.

Unlike Fuchsian triangle groups, the complex angles $\{\theta_1, \theta_2, \theta_3\}$ do not suffice to determine $\Delta(R_1, R_2, R_3)$ up to $\text{Isom}(\mathbf{H}_{\mathbb{C}}^2)$ -equivalence. We also need to consider Cartan's *angular invariant*

$$\psi = \arg(h(v_1, v_2)h(v_2, v_3)h(v_3, v_1)). \quad (4)$$

A complex triangle is uniquely determined up to complex hyperbolic isometry by the complex angles between the walls, and the angular invariant. See [1] and [15, Proposition 1]. Up to the action of complex conjugation on $\mathbf{H}_{\mathbb{C}}^2$, we can assume $\psi \in [0, \pi]$.

We call the angular invariant *rational* if $\psi = s\pi/t$ for some (relatively prime) $s, t \in \mathbb{Z}$. In other words, the angular invariant is rational if and only if $e^{i\psi}$ is a root of unity.

Let $\Delta(R_1, R_2, R_3)$ be a complex hyperbolic triangle group in $U(2, 1)$ with reflection factors η_j , complex angles θ_j , polar vectors v_j , $j = 1, 2, 3$, and angular invariant ψ . Suppose that $\{v_1, v_2, v_3\}$ is a basis for V . Then $\Delta(R_1, R_2, R_3)$ preserves the hermitian form

$$h_{\Delta(R_1, R_2, R_3)} = \begin{pmatrix} 1 & e^{i\psi} \cos \theta_1 & e^{i\psi} \cos \theta_3 \\ e^{-i\psi} \cos \theta_1 & 1 & e^{i\psi} \cos \theta_2 \\ e^{-i\psi} \cos \theta_3 & e^{-i\psi} \cos \theta_2 & 1 \end{pmatrix}. \quad (5)$$

We denote this by h_{Δ} when the generators are clear.

3 Arithmetic subgroups of $U(2, 1)$

Let F be a totally real number field, E a totally imaginary quadratic extension, and \mathcal{D} a central simple E -algebra of degree d . Let $\tau : \mathcal{D} \rightarrow \mathcal{D}$ be an involution, that is, an antiautomorphism of order two. Then τ is of *second kind* if $\tau|_E$ is the Galois involution of E/F . There are two cases of interest.

1. If $\mathcal{D} = E$ (i.e., $d = 1$), then τ is the Galois involution.
2. If $d = 3$, then \mathcal{D} is a cubic division algebra with center E .

See [9] for more on algebras with involution.

For d as above, let $r = 3/d$. A form $h : \mathcal{D}^r \rightarrow \mathcal{D}$ is called *hermitian* or τ -*hermitian* if it satisfies the usual definition of a hermitian form with τ in place of complex conjugation. If $d = 1$, then h is a hermitian form on E^3 as usual. If $d = 3$, then there exists an element $x \in \mathcal{D}^*$ such that $\tau(x) = x$ and $h(y_1, y_2) = \tau(y_1)xy_2$ for all $y_1, y_2 \in \mathcal{D}$.

This determines an algebraic group \mathcal{G} , the group of elements in $\mathrm{GL}_r(\mathcal{D})$ preserving h . For every embedding $\iota : F \rightarrow \mathbb{R}$, we obtain an embedding of \mathcal{G} into the real Lie group $\mathrm{U}(\iota(h))$. Let $\overline{\mathcal{G}}$ be the associated projective unitary group.

If \mathcal{O} is an order in \mathcal{D}^r , then the subgroup $\Gamma_{\mathcal{O}}$ of $\mathrm{GL}_r(\mathcal{O})$ preserving h embeds as a discrete subgroup of

$$\mathcal{G}(\mathbb{R}) = \prod_{\iota: F \rightarrow \mathbb{R}} \mathrm{U}(\iota(h)).$$

If $\overline{\Gamma}_{\mathcal{O}}$ is the image of $\Gamma_{\mathcal{O}}$ in $\overline{\mathcal{G}}$, then $\overline{\Gamma}_{\mathcal{O}}$ is a discrete subgroup of the associated product of projective unitary groups.

The projection of $\Gamma_{\mathcal{O}}$ onto any factor of $\mathcal{G}(\mathbb{R})$ is discrete if and only if the kernel of the projection of $\mathcal{G}(\mathbb{R})$ onto that factor is compact. Therefore, we obtain a discrete subgroup of $\mathrm{U}(2, 1)$ if and only if $\mathrm{U}(\iota(h))$ is noncompact for exactly one real embedding of F .

Then $\overline{\Gamma}_{\mathcal{O}}$ is a lattice in $\mathrm{PU}(2, 1)$ by the well-known theorem of Borel and Harish-Chandra. An arithmetic lattice in $\mathrm{PU}(2, 1)$ is any lattice $\Gamma < \mathrm{PU}(2, 1)$ which is commensurable with $\overline{\Gamma}_{\mathcal{O}}$ for some \mathcal{G} as above and an order \mathcal{O} in \mathcal{D} .

Since arithmeticity only requires commensurability with $\Gamma_{\mathcal{O}}$, studying an arbitrary Γ in the commensurability class of $\Gamma_{\mathcal{O}}$ requires great care. The image of any element $\gamma \in \Gamma$ in $\mathrm{PU}(2, 1)$ does, however, have a representative in $\mathrm{GL}_3(E)$, that is, if there exists $x \in \mathbb{C}^\times$ so $x\gamma \in \mathrm{GL}_3(E)$. This follows from the fact, due to Vinberg [19], that Γ is F -defined over the *adjoint form* $\overline{\mathcal{G}}$, i.e.,

$$\mathbb{Q}(\mathrm{Tr Ad } \Gamma) = F.$$

This important fact also follows from [14, Proposition 4.2].

4 Proofs of Theorems 1.3 and 1.4

We require some elementary results from the theory of discrete subgroups of Lie groups. The primary reference is [16]. Let G be a second countable, locally compact group and $\Gamma < G$ a lattice. Recall that G/Γ carries a

finite G -invariant measure and Γ is *uniform* in G if G/Γ is compact. For a subgroup $H < G$, we let $Z_G(H)$ denote the centralizer of H in G . We need the following two results.

Lemma 4.1 ([16] Lemma 1.14). *Let G be a second countable locally compact group, $\Gamma < G$ a lattice, $\Delta \subset \Gamma$ a finite subset, and $Z_G(\Delta)$ the centralizer of Δ in G . Then, $Z_G(\Delta)\Gamma$ is closed in G .*

Theorem 4.2 ([16] Theorem 1.13). *Let G be a second countable locally compact group, $\Gamma < G$ be a uniform lattice, and $H < G$ be a closed subgroup. Then $H\Gamma$ is closed in G if and only if $H \cap \Gamma$ is a lattice in H .*

We now use the above to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that Γ is a cocompact arithmetic lattice in $U(2, 1)$ containing a complex reflection and that Δ is the subgroup of Γ generated by this reflection. The centralizer of Δ in $U(2, 1)$ is isomorphic to the extension of $U(1, 1)$ by the center of $U(2, 1)$, and is the stabilizer in $U(2, 1)$ of the wall of the reflection that generates Δ . It follows from Lemma 4.1 and Theorem 4.2 that $\Gamma \cap U(1, 1)$ is a lattice. Since any sublattice of an arithmetic lattice is arithmetic, Γ contains a totally geodesic arithmetic Fuchsian subgroup. \square

Proof of Theorem 1.4. A totally geodesic arithmetic Fuchsian group comes from a subalgebra of \mathcal{D}^r , with notation as in §3. When Γ is of second type, \mathcal{D} is a cubic division algebra. The totally geodesic Fuchsian group would correspond to a quaternion subalgebra of \mathcal{D} , which is impossible. When Γ is of first type, this quaternion subalgebra corresponds to rank 2 subspaces of E^3 on which h has signature $(1, 1)$. Therefore, Γ contains complex reflections if and only if Γ is of first type. \square

Remark. One can also prove Theorem 1.4 using the structure of unit groups of division algebras.

We now briefly describe how these results generalize to reflections acting on higher-dimensional complex hyperbolic spaces. If $\Gamma < U(n, 1)$ is a lattice, an element $R \in \Gamma$ is a *codimension s reflection* if it stabilizes a totally geodesic embedded $\mathbf{H}_{\mathbb{C}}^{n-s}$ and acts by an element of the unitary group of the normal bundle to the wall. If Γ is arithmetic, the associated algebraic group is constructed via a hermitian form on \mathcal{D}^r , where \mathcal{D} is a division algebra of degree d with involution of the second kind over a totally imaginary field E , and where $rd = n + 1$.

Theorem 4.3. *Suppose $\Gamma < \mathrm{U}(n, 1)$ is a cocompact arithmetic lattice with associated algebraic group coming from a hermitian form on \mathcal{D}^r , where \mathcal{D} is a central simple algebra with involution of the second kind. If Γ contains a codimension s reflection, then Γ contains a cocompact lattice in $\mathrm{U}(n - s, 1)$. Also, $n - s + 1 = \ell d$ for some $1 < \ell \leq r$ and the associated algebraic subgroup comes from a hermitian form on \mathcal{D}^ℓ .*

Corollary 4.4. *Let $\Gamma < \mathrm{U}(n, 1)$ be an arithmetic lattice generated by complex reflections through totally geodesic walls isometric to $\mathbf{H}_{\mathbb{C}}^{n-1}$. Then Γ is of so-called first type, i.e., the associated algebraic group is the automorphism group of a hermitian form on E^{n+1} , where E is some totally imaginary quadratic extension of a totally real field.*

5 Arithmetic data for complex hyperbolic triangle groups

In this section, we relate the geometric invariants of a complex triangle to the arithmetic invariants of the complex reflection group. It is the technical heart of the paper.

Let $\Gamma = \triangle(R_1, R_2, R_3)$ be a complex hyperbolic triangle group with reflection factors η_j , complex angles θ_j , and angular invariant ψ . Assume that Γ is an arithmetic lattice in $\mathrm{U}(2, 1)$. By Theorem 1.4, Γ is of first type, so there is an associated hermitian form h over a totally imaginary field E . Let F be the totally real quadratic subfield of E .

Lemma 5.1. *We can choose polar vectors v_j for the reflection R_j so that $v_j \in E^3$.*

Proof. Associated with each reflection is an arithmetic Fuchsian subgroup of Γ . When Γ is a uniform lattice, this follows from Theorem 1.3. For the nonuniform case, see [6, Chapter 5]. Arithmetic Fuchsian subgroups stabilizing a complex hyperbolic line come from the h -orthogonal complement of an h -positive line in E^3 . (To be more precise, this line is h -positive over the unique real embedding of F at which h is indefinite.) Any vector in E^3 representing this line is a polar vector for R_j . \square

This leads us to the following important fact.

Lemma 5.2. *Each reflection factor η_j is contained in E .*

Proof. It follows from Proposition 4.2 in [14] that there exists an $x_j \in \mathbb{C}$ so that $x_j R_j \in \mathrm{GL}_3(E)$ (see the end of §3 above). By Lemma 5.1, and because the h -orthogonal complement to a polar vector evidently has an E -basis, E^3 has a basis of eigenvectors for R_j . The lemma follows from Lemma 2.1. \square

Now we turn to the complex angles and the angular invariant.

Lemma 5.3. *For each j , $\cos^2 \theta_j \in F$ and $e^{2i\psi} \in E$.*

Proof. Choose polar vectors $v_j \in E^3$. The terms in Equations (2) and (3) resulting from these choices of polar vectors are all contained in E . Hence $\cos^2 \theta_j \in F$. One can also prove this using $\mathrm{Tr} \mathrm{Ad}(R_1 R_2)$ and Lemma 5.2.

Similarly, consider

$$\delta = h(v_1, v_2)h(v_2, v_3)h(v_3, v_1) = r e^{i\psi} \in E$$

from Equation (4). Note that $e^{i\psi} \in E$ if and only if $r \in E$. Either way, when $\delta \neq 0$, we have $\delta/\bar{\delta} = e^{2i\psi} \in E$. This completes the proof. \square

Combining the above, we see that

$$\mathbb{Q}(\eta_1, \eta_2, \eta_3, \cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, e^{2i\psi}) \subseteq E.$$

We can also bound E from above using the fact that $E \subseteq \mathbb{Q}(\mathrm{Tr} \Gamma)$. Using well-known computations of traces for products of reflections (e.g., [10, §4] or [15]), we have

$$\mathbb{Q}(\mathrm{Tr} \Gamma) = \mathbb{Q}(\eta_1, \eta_2, \eta_3, \cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, e^{i\psi} \cos \theta_1 \cos \theta_2 \cos \theta_3).$$

Similarly,

$$\mathbb{Q}(\mathrm{Re}(\eta_1), \mathrm{Re}(\eta_2), \mathrm{Re}(\eta_3), \cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, \cos^2 \psi) \subseteq F \subseteq$$

$$\mathbb{Q}(\mathrm{Re}(\eta_1), \mathrm{Re}(\eta_2), \mathrm{Re}(\eta_3), \cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, \cos \psi \cos \theta_1 \cos \theta_2 \cos \theta_3).$$

This gives the following.

Corollary 5.4. *Let Γ be a complex hyperbolic triangle group and an arithmetic lattice in $\mathrm{U}(2, 1)$. If the angular invariant of the triangle associated with Γ is rational, then the fields that define Γ as an arithmetic lattice are subfields of a cyclotomic field.*

Let h_Δ be as in (5) and consider h_Δ as a hermitian form on the extension

$$E_\Delta = \mathbb{Q}(\eta_1, \eta_2, \eta_3, \cos \theta_1, \cos \theta_2, \cos \theta_3, e^{i\psi}),$$

of E . It follows from [10, §2] that h and h_Δ are equivalent over E_Δ . Consequently, h_Δ is indefinite over exactly one complex conjugate pair of places of E . This implies that there are precisely $[E_\Delta : E]$ conjugate pairs of places of E_Δ over which h_Δ is indefinite.

Let H be a hermitian in 3 variables over the complex numbers for which there is a vector with positive H -norm. Then H is indefinite if and only if $\det(H) < 0$. Since any polar vector has positive h_Δ -norm by definition, we have the following.

Proposition 5.5. *There are exactly $[E_\Delta : E]$ complex conjugate pairs of Galois automorphisms τ of $E_\Delta \subset \mathbb{C}$ under which $\tau(\det(h_\Delta))$ is negative. All such automorphisms act trivially on E .*

This has the following consequence for the relationship between the geometry of the triangle and the arithmetic of the lattice.

Corollary 5.6. *If Γ is a complex hyperbolic triangle group and an arithmetic lattice, then the reflection factors of Γ are restricted by the geometry of the triangle. In particular,*

$$E_\Delta = \mathbb{Q}(\cos \theta_1, \cos \theta_2, \cos \theta_3, e^{i\psi}).$$

Proof. Since $\det(h_\Delta)$ is independent of the reflection factors, for each Galois automorphism of

$$E_\Delta/\mathbb{Q}(\cos \theta_1, \cos \theta_2, \cos \theta_3, e^{i\psi})$$

we obtain a new complex conjugate pair of embeddings of E_Δ into \mathbb{C} such that $\det(h_\Delta)$ is negative. Any such automorphism necessarily acts nontrivially on some reflection factor η_j . These embeddings of E_Δ lie over different places of E by Lemma 5.2. This contradicts Proposition 5.5. \square

We also obtain the following dependence between the angular invariant and the angles of the triangle.

Proposition 5.7. *If Γ is a complex hyperbolic triangle group and an arithmetic lattice. If Γ has rational angular invariant and $\theta_j \leq \pi/3$ for $j = 1, 2, 3$, then*

$$\cos^2 \psi \in F' = \mathbb{Q}(\cos^2 \theta_1, \cos^2 \theta_2, \cos^2 \theta_3, \cos \theta_1 \cos \theta_2 \cos \theta_3).$$

Proof. If ψ is rational, then E_Δ is a subfield of a cyclotomic field $K_N = \mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N^{th} root of unity. Therefore the Galois automorphisms of E_Δ are induced by $\zeta_N \mapsto \zeta_N^m$ for some m relatively prime to N .

Consider the stabilizer S of F' in $\text{Gal}(K_N/\mathbb{Q})$. It acts on the roots of unity in E_Δ as a group of rotations along with complex conjugation. By definition of E_Δ , every nontrivial element of S acts nontrivially on $e^{i\psi}$. In particular, if $\cos^2 \psi \notin \mathbb{Q}$ and S contains a rotation through an angle other than an integral multiple of π , then the orbit of $e^{i\psi}$ under S contains two non-complex conjugate points with distinct negative real parts.

Let τ be any such automorphism of E_Δ . Then, since $\tau(\cos \theta_j) = \cos \theta_j$ for all j by definition of S ,

$$\tau(\det(h_\Delta)) = 1 - \sum_{j=1}^3 \cos^2 \theta_j + 2\tau(\cos \psi) \prod_{j=1}^3 \cos \theta_j.$$

Furthermore, $1 - \sum \cos^2 \theta_j \leq 0$ for any triple of angles $\theta_j = \pi/r_j$ that are the angles of a hyperbolic triangle with each $r_j \geq 3$. Since $\tau(\cos \psi) < 0$ and $\cos \theta_j > 0$, it follows that $\tau(\det h_\Delta) < 0$. Since τ acts nontrivially on $e^{2i\psi} \in E$, this contradicts Proposition 5.5. Therefore, S is generated by complex conjugation and rotation by π , so $\cos^2 \psi \in F'$. \square

Remark. For several of Mostow's lattices in [10], $F' = F$ (with notation as above) and $\cos \psi \notin F'$. Thus Proposition 5.7 is the strongest possible constraint on rational angular invariants.

6 Finiteness results

We are now prepared to collect facts from §5 to prove Theorem 1.1. A more precise version is the following.

Theorem 6.1. *Suppose that Γ is a complex hyperbolic triangle group and a nonuniform arithmetic lattice in $\text{U}(2,1)$. Then*

1. *Each generator has order 2, 3, 4, or 6.*
2. *Each complex angle θ_j of the triangle comes from the set*

$$\{\pi/2, \pi/3, \pi/4, \pi/6, 0\}.$$

3. *If ψ is the angular invariant, then $e^{i\psi}$ lies in a biquadratic extension of \mathbb{Q} .*

4. If ψ is rational, then $\psi = s\pi/t$ for

$$t \in \{2, 3, 4, 6, 8, 12\}.$$

Proof. Since Γ is a nonuniform arithmetic lattice, the associated field E is imaginary quadratic. For 1, we apply Lemma 5.2 to E . For 2 and 3, we apply Lemma 5.3. Then 4 follows from determining those integers m so that $\varphi(m) = 2$ or 4 and $e^{2i\psi}$ is at most quadratic over \mathbb{Q} , where φ is Euler's totient function. \square

See [13, 3] for the known nonuniform arithmetic complex hyperbolic triangle groups. We now determine the right triangle groups that can determine an arithmetic lattice in $\mathrm{SU}(2, 1)$.

Proof of Theorem 1.6. Suppose that Γ is an arithmetic complex hyperbolic triangle group with $\theta_1 = \pi/2$. The hermitian form h_Δ associated with the triangle has determinant

$$1 - \cos^2 \theta_2 - \cos^2 \theta_3.$$

By Lemma 5.3, this is an element of the totally real field F that defines Γ as an arithmetic lattice. Consequently, there is no Galois automorphism of F over \mathbb{Q} under which this expression remains negative.

This is precisely Takeuchi's condition that determines whether or not the triangle in the hyperbolic plane with angles $\pi/2, \theta_2, \theta_3$ determines an arithmetic Fuchsian group. The theorem follows from Takeuchi's classification of arithmetic Fuchsian right triangle groups, Lemma 5.3, and Corollary 5.6 \square

There are 41 such right triangles in \mathbf{H}^2 . We now finish the paper with finiteness for arithmetic complex hyperbolic triangle groups with equilateral complex triangle and rational angular invariant.

Proof of Theorem 1.7. Let Γ be an arithmetic complex hyperbolic triangle group with equilateral triangle of angles π/n and angular invariant ψ . By Proposition 5.7, we can assume that $\psi = s\pi/12n$ for some integer s . Indeed, $F' = \mathbb{Q}(\cos \pi/n)$, and the assertion follows from an easy Galois theory computation.

Then

$$\det(h_\Delta) = 1 - 3\cos^2(\pi/n) + 2\cos(s\pi/12n)\cos^3(\pi/n), \quad (6)$$

so we want to find a nontrivial Galois automorphism of F_Δ whose restriction to F is nontrivial and such that the image of (6) under this automorphism is negative. Let p be the smallest rational prime not dividing $12n$. This determines a nontrivial Galois automorphism τ_p of F_Δ under which

$$\tau_p(\det(h_\Delta)) = 1 - 3\cos^2(p\pi/n) + 2\cos(ps\pi/12n)\cos^3(p\pi/n). \quad (7)$$

It is nontrivial on F by definition. If we show that $\tau_p(\det(h_\Delta)) < 0$ for n sufficiently large, this, along with Corollary 5.6, suffices to prove the theorem.

First, notice that the function

$$D(x, y) = 1 - 3\cos^2 x + 2\cos y \cos^3 x$$

is an increasing function of $x \in (0, \pi/2)$ for any fixed y . In our language, this implies that if y is the angular invariant of an equilateral complex triangle in $\mathbf{H}_\mathbb{C}^2$ with angle x , then it remains an angular invariant for a complex triangle with angle x' for any $x' < x$. Similarly, if we know that $\pi/12n$ is an angular invariant for a triangle with angles $p\pi/n$, then we know that $ps\pi/n$ (more precisely, a representative modulo 2π) is the angular invariant of an equilateral triangle in $\mathbf{H}_\mathbb{C}^2$ with angles $p\pi/n$. Therefore, it is enough to show that $\pi/12n$ is the angular invariant of a triangle having angles $p\pi/n$ for all sufficiently large n , where p is the smallest not prime dividing $12n$.

From the above, we conclude further that it suffices to show that there exists a function $q(n)$ such that $p < q(n)$ and

$$1 - 3\cos^2(q(n)\pi/n) + 2\cos(\pi/12n)\cos^3(q(n)\pi/n) < 0 \quad (8)$$

for all sufficiently large n . To prove this, we consider the function $j(n)$, defined by Jacobsthal [8]. For any integer n , $j(n)$ is the smallest integer such that any $j(n)$ consecutive integers must contain one that is relatively prime to n . Clearly $p \leq j(12n)$.

Iwaniec [7] proved that

$$j(n) \ll (\log n)^2.$$

Therefore, for any $\epsilon > 0$, there is an n_ϵ so that the first prime number coprime to $12n$ is at most $\log(12n)^{2+\epsilon}$ for every $n \geq n_\epsilon$. Consider the function

$$f_\epsilon(x) = 1 - 3\cos^2(\log(12/x)^{2+\epsilon}\pi x) + 2\cos(\pi x/12)\cos^3(\log(12/x)^{2+\epsilon}\pi x).$$

Then $\lim_{x \rightarrow 0} f_\epsilon(x)$ exists and equals 0 for all $\epsilon > 0$. Further, $x = 0$ is a local maximum of f_ϵ , so $f_\epsilon(1/n) < 0$ for all sufficiently large n .

Taking $q(n) = \log(n)^{2+\epsilon}$ for any small ϵ shows that (8) holds for all sufficiently large n . This implies that (7) is negative for all large n . This proves the theorem. \square

Unfortunately, the proof of Theorem 1.7 isn't effective, so we cannot list the angles that can possibly determine an arithmetic lattice. In particular, we don't know which n makes the bound from [7] effective for any $\epsilon > 0$. If this bound is less than $n = 10^5$ for some ϵ , which computer experiments show is extraordinarily likely, then we obtain $n < 5,000,000$. We expect the actual bound to be quite a bit smaller, especially given that the smallest equilateral triangle in \mathbf{H}^2 that defines an arithmetic Fuchsian group has angles $\pi/15$.

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